Duality and Spherical Adjunction from Microlocalization

joint work with Christopher Kuo

Wenyuan Li

Northwestern

Homological Mirror Symmetry

- Mirror symmetry is a phenomenon for a pair of Calabi-Yau manifolds (a complex manifold with a Kähler metric and a complex volume form) X and $X^\vee$ where the symplectic invariants of X matches with the complex invariants of $X^\vee$ and vise versa.

Homological Mirror Symmetry

- Mirror symmetry is a phenomenon for a pair of Calabi-Yau manifolds (a complex manifold with a Kähler metric and a complex volume form) X and $X^\vee$ where the symplectic invariants of X matches with the complex invariants of $X^\vee$ and vise versa.
- There are different versions of mirror symmetry. For example, one can compare two series of numbers or generating functions (enumerative mirror symmetry), two versions of variations of Hodge structures (Hodge theoretic mirror symmetry), or even two A_∞-categories (homological mirror symmetry).

Homological Mirror Symmetry

- Given a complex manifold, one can consider a dg category called the derived category of coherent sheaves, whose objects are chain complexes of coherent sheaves (generalization of complexes of holomorphic vector bundles).

Homological Mirror Symmetry

- Given a complex manifold, one can consider a dg category called the derived category of coherent sheaves, whose objects are chain complexes of coherent sheaves (generalization of complexes of holomorphic vector bundles).
- Given a symplectic manifold that satisfies certain assumptions, one can consider an A_∞-category called the Fukaya category, whose objects are Lagrangian submanifolds (half dimensional submanifolds where the symplectic form vanish), and whose morphisms are linear combinations of intersection points between Lagrangians with differential given by counting pseudo-holomorphic curves.

Homological Mirror Symmetry

- Given a complex manifold, one can consider a dg category called the derived category of coherent sheaves, whose objects are chain complexes of coherent sheaves (generalization of complexes of holomorphic vector bundles).
- Given a symplectic manifold that satisfies certain assumptions, one can consider an A_∞-category called the Fukaya category, whose objects are Lagrangian submanifolds (half dimensional submanifolds where the symplectic form vanish), and whose morphisms are linear combinations of intersection points between Lagrangians with differential given by counting pseudo-holomorphic curves.
- Homological mirror symmetry says that for a certain pair of Calabi-Yau manifolds X and $X^\vee$ the derived Fukaya category of X is equivalent to the derived categories of coherent sheaves on $X^\vee$ and vise versa.

Homological Mirror Symmetry

- Homological mirror symmetry can be generalized to non Calabi-Yau manifolds as well. Auroux suggested that a Fano manifold X with an anticanonical divisor D (such that $D = -K_X$) is mirror to a Landau-Ginzburg model, i.e. a Kähler manifold $X^\vee$ with a holomorphic function called superpotential $W^\vee : X^\vee \to \mathbb{C}$.

Homological Mirror Symmetry

- Homological mirror symmetry can be generalized to non Calabi-Yau manifolds as well. Auroux suggested that a Fano manifold X with an anticanonical divisor D (such that $D = -K_X$) is mirror to a Landau-Ginzburg model, i.e. a Kähler manifold $X^\vee$ with a holomorphic function called superpotential $W^\vee : X^\vee \to \mathbb{C}$.
- We should expect the following equivalences of categories

$$D^b Coh(X) \simeq D^\pi WFuk(X^\vee, W^\vee),$$
$$D^b Coh(D) \simeq D^\pi WFuk((W^\vee)^{-1}(1)),$$
$$D^b Coh(X \setminus D) \simeq D^\pi WFuk(X^\vee).$$

Homological Mirror Symmetry

- Homological mirror symmetry can be generalized to non Calabi-Yau manifolds as well. Auroux suggested that a Fano manifold X with an anticanonical divisor D (such that $D = -K_X$) is mirror to a Landau-Ginzburg model, i.e. a Kähler manifold $X^\vee$ with a holomorphic function called superpotential $W^\vee : X^\vee \to \mathbb{C}$.
- We should expect the following equivalences of categories

$$D^b Coh(X) \simeq D^\pi WFuk(X^\vee, W^\vee),$$
$$D^b Coh(D) \simeq D^\pi WFuk((W^\vee)^{-1}(1)),$$
$$D^b Coh(X \setminus D) \simeq D^\pi WFuk(X^\vee).$$

- On the left hand side, there are different functors that relate different categories of coherent sheaves. Our goal is to understand the structures on the right hand side.

Duality and Spherical Adjunction

- Let X be a complex manifold and D be a divisor. Let $i : D \hookrightarrow X$ be the inclusion. The pull back functor i^* has both adjoints $i_*, i_!$ (where $i_! = i_*(-) \otimes \mathcal{O}_X(D)[-1]$). There are exact triangles

$$- \otimes \mathcal{O}_X(-D) \to \mathrm{id} \to i_* i^*, \ i_! i^* \to \mathrm{id} \to - \otimes \mathcal{O}_X(D),$$

Here $- \otimes \mathcal{O}_X(\pm D)$ (the cotwist and dual cotwist) are inverse autoequivalences. There are also exact triangles

$$i^* i_* \to \mathrm{id} \to - \otimes i^* \mathcal{O}_X(-D)[1], \ - \otimes i^* \mathcal{O}_X(D)[-1] \to \mathrm{id} \to i^* i_!.$$

Here $- \otimes i^* \mathcal{O}_X(\pm D)$ (the twist and dual twist) are also inverse autoequivalences. We call i^* a spherical functor.

Duality and Spherical Adjunction

- Let X be a complex manifold and D be a divisor. Let $i : D \hookrightarrow X$ be the inclusion. The pull back functor i^* has both adjoints $i_*, i_!$ (where $i_! = i_*(-) \otimes \mathcal{O}_X(D)[-1]$). There are exact triangles

$$- \otimes \mathcal{O}_X(-D) \to \mathrm{id} \to i_* i^*, \ i_! i^* \to \mathrm{id} \to - \otimes \mathcal{O}_X(D),$$

  Here $- \otimes \mathcal{O}_X(\pm D)$ (the cotwist and dual cotwist) are inverse autoequivalences. There are also exact triangles

$$i^* i_* \to \mathrm{id} \to - \otimes i^* \mathcal{O}_X(-D)[1], \ - \otimes i^* \mathcal{O}_X(D)[-1] \to \mathrm{id} \to i^* i_!.$$

  Here $- \otimes i^* \mathcal{O}_X(\pm D)$ (the twist and dual twist) are also inverse autoequivalences. We call i^* a spherical functor.
- When X is a Fano manifold and D is an anticanonical divisor, we know that $- \otimes \mathcal{O}_X(D)$ is the inverse Serre functor.

Duality and Spherical Adjunction

Theorem (Abouzaid-Ganatra, unpublished)

Let $W : X \to \mathbb{C}$ *be an (exact symplectic) Landau-Ginzburg model. Then there is a cap functor* $\cap : Fuk(X, W) \to Fuk(W^{-1}(1))$ *that is spherical.*

Duality and Spherical Adjunction

Theorem (Abouzaid-Ganatra, unpublished)

Let $W : X \to \mathbb{C}$ be an (exact symplectic) Landau-Ginzburg model. Then there is a cap functor $\cap : Fuk(X, W) \to Fuk(W^{-1}(1))$ that is spherical.

Definition (Sylvan, 2019)

Let X be an exact symplectic manifold with contact boundary. Then a symplectic hypersurface F in the contact boundary is called swappable if there is some positive Hamiltonian flow φ_t that sends F to itself such that $\varphi_t(F) \cap F = \varnothing$ when $t \neq 0, 1$.

Duality and Spherical Adjunction

Theorem (Abouzaid-Ganatra, unpublished)

Let $W : X \to \mathbb{C}$ be an (exact symplectic) Landau-Ginzburg model. Then there is a cap functor $\cap : Fuk(X, W) \to Fuk(W^{-1}(1))$ that is spherical.

Definition (Sylvan, 2019)

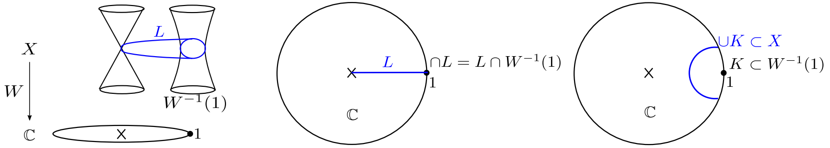
Let X be an exact symplectic manifold with contact boundary. Then a symplectic hypersurface F in the contact boundary is called swappable if there is some positive Hamiltonian flow φ_t that sends F to itself such that $\varphi_t(F) \cap F = \varnothing$ when $t \neq 0, 1$.

Theorem (Sylvan, 2019)

Let X be an exact symplectic manifold with contact boundary and F be an exact swappable symplectic hypersurface. Then there is a cup functor $\cup : WFuk(F) \to WFuk(X, F)$ that is spherical.

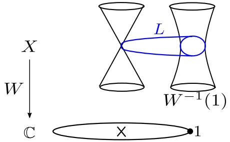
Duality and Spherical Adjunction

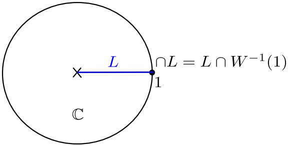
- Objects in $Fuk(X, W)$ are noncompact exact Lagrangians (with cylindrical ends) that end in the fiber $W^{-1}(1)$, and morphisms are given by Floer cohomology. $\cap$ sends the noncompact Lagrangian L to its intersection with the fiber $L \cap W^{-1}(1)$.

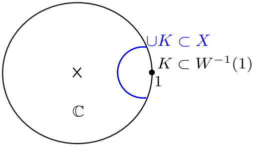


Duality and Spherical Adjunction

- Objects in $Fuk(X, W)$ are noncompact exact Lagrangians (with cylindrical ends) that end in the fiber $W^{-1}(1)$, and morphisms are given by Floer cohomology. $\cap$ sends the noncompact Lagrangian L to its intersection with the fiber $L \cap W^{-1}(1)$.
- Objects in $WFuk(X, W)$ or $WFuk(X, F)$ are noncompact exact Lagrangians (with cylindrical ends) that avoid the fiber $W^{-1}(1)$, and morphisms are given by Floer cohomology with wrappings at infinity that avoids the fiber. $\cup$ sends a Lagrangian in the fiber $W^{-1}(1)$ or F to a noncompact Lagrangian by parallel transport along a U-shape path that goes around $W^{-1}(1)$ or F.

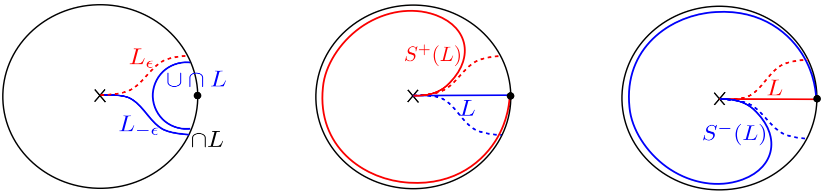






Duality and Spherical Adjunction

- Under some assumptions, we should identify *Fuk*(X, W) with (part of) *WFuk*(X, W), so that $\cap$ and $\cup$ form an adjunction.

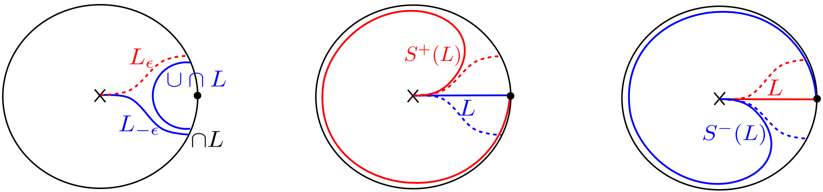


Duality and Spherical Adjunction

- Under some assumptions, we should identify $Fuk(X, W)$ with (part of) $WFuk(X, W)$, so that $\cap$ and $\cup$ form an adjunction.
- We should expect exact triangles of functors in Fukaya categories

$$\cup^l \cap \to \mathrm{id} \to S^+,\ S^- \to \mathrm{id} \to \cup^r \cap\,.$$

$\cup^l$ is given by pushing the noncompact Lagrangian to the fiber $W^{-1}(1)$ counterclockwise, and $\cup^r$ is given by pushing the noncompact Lagrangian to the fiber clockwise.



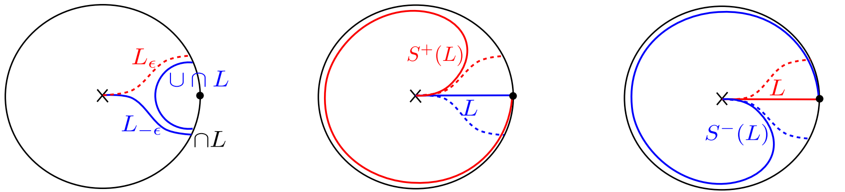
Duality and Spherical Adjunction

- Under some assumptions, we should identify $Fuk(X, W)$ with (part of) $WFuk(X, W)$, so that $\cap$ and $\cup$ form an adjunction.
- We should expect exact triangles of functors in Fukaya categories

$$\cup^l \cap \to \mathrm{id} \to S^+,\ S^- \to \mathrm{id} \to \cup^r \cap .$$

$\cup^l$ is given by pushing the noncompact Lagrangian to the fiber $W^{-1}(1)$ counterclockwise, and $\cup^r$ is given by pushing the noncompact Lagrangian to the fiber clockwise.

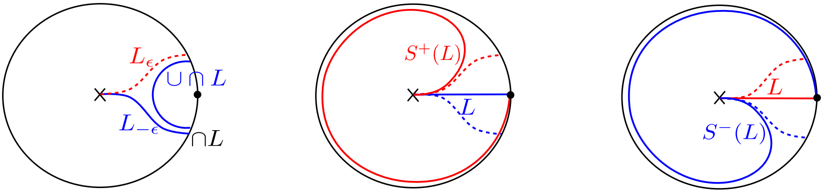
- Then the (dual) cotwist $S^\pm$ are autoequivalences given by wrapping the Lagrangian around once (counter)clockwise.



Duality and Spherical Adjunction

Conjecture (Kontsevich, Seidel)

The dual cotwist $S^+ : Fuk(X, W) \to Fuk(X, W)$ given by wrapping around once is the inverse Serre functor on $Fuk(X, W)$.



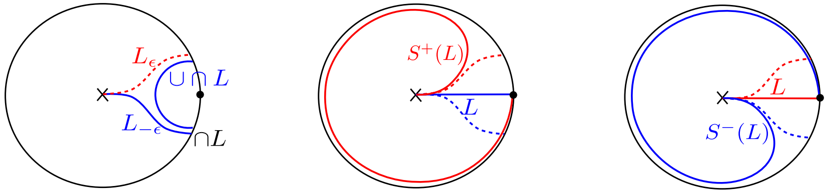
Duality and Spherical Adjunction

Conjecture (Kontsevich, Seidel)

The dual cotwist $S^+ : Fuk(X, W) \to Fuk(X, W)$ given by wrapping around once is the inverse Serre functor on $Fuk(X, W)$.

Remark

Seidel has obtained a series of results in this direction. A complete proof of the conjecture may appear in the work in progress by Bai-Seidel.



Sheaves and Floer theory invariants

- Nadler-Zaslow noticed that the Fukaya category of the cotangent bundle is closely related to constructible sheaves on the manifold.

Sheaves and Floer theory invariants

- Nadler-Zaslow noticed that the Fukaya category of the cotangent bundle is closely related to constructible sheaves on the manifold.
- A constructible sheaf on a manifold with respect to a stratification is a restricts to local systems on all strata. For a stratification, one can consider its conormal bundle, which defines a singular conical Lagrangian in the cotangent bundle.

Sheaves and Floer theory invariants

- Nadler-Zaslow noticed that the Fukaya category of the cotangent bundle is closely related to constructible sheaves on the manifold.
- A constructible sheaf on a manifold with respect to a stratification is a restricts to local systems on all strata. For a stratification, one can consider its conormal bundle, which defines a singular conical Lagrangian in the cotangent bundle.
- Nadler-Zaslow and Nadler (2006) showed that the category of all constructible sheaves $Sh^b_{\text{con}}(M)$ is equivalent to a infinitesimally wrapped Fukaya categories $Fuk_\epsilon(T^*M)$.

Sheaves and Floer theory invariants

- For any constructible sheaf, there is a more refined notion called singular support (typically a smaller subset in the conormal bundle of the stratification), which also defines a singular conical Lagrangian in the cotangent bundle.

Sheaves and Floer theory invariants

- For any constructible sheaf, there is a more refined notion called singular support (typically a smaller subset in the conormal bundle of the stratification), which also defines a singular conical Lagrangian in the cotangent bundle.
- For the cotangent bundle T^*M, consider an exact symplectic hypersurface F at the contact boundary $T^{*,\infty}M$. We assume that it has a Lagrangian sksleton Λ_F (it is a deformation retract of F that satisfies certain conditions).

Sheaves and Floer theory invariants

- For any constructible sheaf, there is a more refined notion called singular support (typically a smaller subset in the conormal bundle of the stratification), which also defines a singular conical Lagrangian in the cotangent bundle.
- For the cotangent bundle T^*M, consider an exact symplectic hypersurface F at the contact boundary $T^{*,\infty}M$. We assume that it has a Lagrangian sksleton Λ_F (it is a deformation retract of F that satisfies certain conditions).
- Ganatra-Pardon-Shende (2018) showed the equivalence between the compact objects in (unbounded) category of sheaves with singular support on the skeleton of the fiber Λ_F and the partially wrapped Fukaya category

$$Sh^c_{\Lambda_F}(M) \simeq D^\pi WFuk(T^*M, F)^{\text{op}}.$$

Sheaves and Floer theory invariants

- More generally, for any (polarizable) exact symplectic manifold F with Lagrangian sksleton Λ_F, there is a category of microlocal sheaves $\mu Sh_{\Lambda_F}(\Lambda_F)$ defined by microlocalization (this is done by dg localization plus sheafification of categories).

Sheaves and Floer theory invariants

- More generally, for any (polarizable) exact symplectic manifold F with Lagrangian sksleton Λ_F, there is a category of microlocal sheaves $\mu Sh_{\Lambda_F}(\Lambda_F)$ defined by microlocalization (this is done by dg localization plus sheafification of categories).
- Ganatra-Pardon-Shende (2018) showed the equivalence between the compact objects in (unbounded) category of microlocal sheaves on the skeleton Λ_F and the wrapped Fukaya category plus a commutative diagram

$$\mu Sh^c_{\Lambda_F}(\Lambda_F) \simeq D^\pi WFuk(F)^{\mathrm{op}}.$$

Sheaves and Floer theory invariants

- More generally, for any (polarizable) exact symplectic manifold F with Lagrangian sksleton Λ_F, there is a category of microlocal sheaves $\mu Sh_{\Lambda_F}(\Lambda_F)$ defined by microlocalization (this is done by dg localization plus sheafification of categories).
- Ganatra-Pardon-Shende (2018) showed the equivalence between the compact objects in (unbounded) category of microlocal sheaves on the skeleton Λ_F and the wrapped Fukaya category plus a commutative diagram

$$\mu Sh^c_{\Lambda_F}(\Lambda_F) \simeq D^\pi WFuk(F)^{\mathrm{op}}.$$

- Moreover, there is a microlocalization functor which corresponds to the cap functor $Fuk(X, W) \to Fuk(F)$

$$m_{\Lambda_F} : Sh_{\Lambda_F}(M) \to \mu Sh_{\Lambda_F}(\Lambda_F).$$

Main results

Definition

Let $\Lambda \subset T^{*,\infty}M$ be a subanalytic Legendrian. It is called swappable if there is a positive Hamiltonian that sends the positive push off Λ_ϵ to an arbitrary small neighbourhood of the negative push off $\Lambda_{-\epsilon}$ and vise versa.

Main results

Definition

Let $\Lambda \subset T^{*,\infty}M$ be a subanalytic Legendrian. It is called swappable if there is a positive Hamiltonian that sends the positive push off Λ_ϵ to an arbitrary small neighbourhood of the negative push off $\Lambda_{-\epsilon}$ and vise versa.

Definition

Let $\Lambda \subset T^{*,\infty}M$ be a subanalytic Legendrian. It is called fully stopped if $Sh_\Lambda(M)$ is compactly generated by a proper category.

Main results

Definition

Let $\Lambda \subset T^{*,\infty}M$ be a subanalytic Legendrian. It is called swappable if there is a positive Hamiltonian that sends the positive push off Λ_ϵ to an arbitrary small neighbourhood of the negative push off $\Lambda_{-\epsilon}$ and vise versa.

Definition

Let $\Lambda \subset T^{*,\infty}M$ be a subanalytic Legendrian. It is called fully stopped if $Sh_\Lambda(M)$ is compactly generated by a proper category.

Theorem (Kuo-L.)

When $\Lambda \subset T^{*,\infty}M$ is a compact swappable or full Legendrian stop, the microlocalization functor $m_\Lambda : Sh_\Lambda(M) \to \mu Sh_\Lambda(\Lambda)$ is spherical, and the dual cotwist is the inverse Serre functor on the proper subcategory.

- When we unpack the definition of spherical functors, our result says that $m_\Lambda : Sh_\Lambda(M) \to \mu Sh_\Lambda(\Lambda)$ has both left and right adjoints $m_\Lambda^{l/r}$, such that there are exact triangles of functors on sheaves

$$m_\Lambda^l m_\Lambda \to \mathrm{id} \to S_\Lambda^+,\ S_\Lambda^- \to \mathrm{id} \to m_\Lambda^r m_\Lambda$$

such that the (dual) cotwist $S_\Lambda^\pm$ are inverse autoequivalences, and exact triangles

$$T_\Lambda^+ \to \mathrm{id} \to m_\Lambda m_\Lambda^l,\ m_\Lambda m_\Lambda^r \to \mathrm{id} \to T_\Lambda^-$$

such that the (dual) twist $T_\Lambda^\pm$ are inverse autoequivalences.

- When we unpack the definition of spherical functors, our result says that $m_\Lambda : Sh_\Lambda(M) \to \mu Sh_\Lambda(\Lambda)$ has both left and right adjoints $m_\Lambda^{l/r}$, such that there are exact triangles of functors on sheaves

$$m_\Lambda^l m_\Lambda \to \mathrm{id} \to S_\Lambda^+,\ S_\Lambda^- \to \mathrm{id} \to m_\Lambda^r m_\Lambda$$

  such that the (dual) cotwist $S_\Lambda^\pm$ are inverse autoequivalences, and exact triangles

$$T_\Lambda^+ \to \mathrm{id} \to m_\Lambda m_\Lambda^l,\ m_\Lambda m_\Lambda^r \to \mathrm{id} \to T_\Lambda^-$$

  such that the (dual) twist $T_\Lambda^\pm$ are inverse autoequivalences.
- Moreover, the dual cotwist S_Λ^+ is the inverse Serre functor, i.e. the cotwist S_Λ^- is the Serre functor on the proper subcategory:

$$Hom(F, G) = Hom(G, S_\Lambda^-(F))^\vee[-n].$$

Idea of the proof

- In the proof, we try to understand the microlocalization functor from the following duality and exact sequence. Namely, microlocalization is measured by the difference between a small negative and positive pushoff by the Reeb flow.

Idea of the proof

- In the proof, we try to understand the microlocalization functor from the following duality and exact sequence. Namely, microlocalization is measured by the difference between a small negative and positive pushoff by the Reeb flow.
- When we consider the vertical translation in $T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$, the exact sequence has appeared in for example Y. Ike 2017.

Idea of the proof

- In the proof, we try to understand the microlocalization functor from the following duality and exact sequence. Namely, microlocalization is measured by the difference between a small negative and positive pushoff by the Reeb flow.
- When we consider the vertical translation in $T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$, the exact sequence has appeared in for example Y. Ike 2017.

Theorem (Sato-Sabloff exact triangle)

Let $\Lambda \subset T^{*,\infty}M$ be subanalytic Legendrian. For $\mathscr{F}, \mathscr{G} \in Sh_\Lambda(M)_0$, $Hom(\mathscr{F}, \mathscr{G}_\epsilon) \simeq Hom(\mathscr{F}, \mathscr{G})$ and there is an exact triangle

$$Hom(\mathscr{F}, \mathscr{G}_{-\epsilon}) \to Hom(\mathscr{F}, \mathscr{G}_\epsilon) \to Hom_{\mu Sh_\Lambda}(m_\Lambda \mathscr{F}, m_\Lambda \mathscr{G}) \xrightarrow{+1} .$$

For $\mathscr{F}, \mathscr{G} \in Sh_\Lambda(M)_0$, when $\mathscr{F}, \mathscr{G}$ have perfect stalks, there is a duality

$$Hom(\mathscr{F}, \mathscr{G}_{-\epsilon} \otimes \omega_M)^\vee \simeq Hom(\mathscr{F}, \mathscr{G}_\epsilon).$$

Idea of the proof

- Using the duality exact sequence, we can understand the adjoint of the microlocalization functor using the doubling functor.

Idea of the proof

- Using the duality exact sequence, we can understand the adjoint of the microlocalization functor using the doubling functor.
- When the Reeb flow is the vertical translation in $T^{*,\infty}(M \times \mathbb{R})$, this is obtained by Guillermou 2012.

Theorem (Guillermou doubling functor)

Let $\Lambda \subset T^{*,\infty}M$ be subanalytic Legendrian. Then there is a fully faithful doubling functor

$$w_\Lambda : \mu Sh_\Lambda(\Lambda) \hookrightarrow Sh_{\Lambda_{-\epsilon} \cup \Lambda_\epsilon}(M)$$

with an exact triangle $T_{-\epsilon} \to T_\epsilon \to w_\Lambda \circ m_\Lambda$ which gives the adjoint functors of m_Λ after further wrappings $Sh_{\Lambda_{-\epsilon} \cup \Lambda_\epsilon}(M) \to Sh_\Lambda(M)$.

Discussion of the result

- When Λ is not swappable or fully stopped, we construct examples where the microlocalization functor m_Λ is not spherical.

Discussion of the result

- When Λ is not swappable or fully stopped, we construct examples where the microlocalization functor m_Λ is not spherical.
- The result on spherical functor and Serre duality is closely related to relative Calabi-Yau structure for the pair $m_\Lambda : Sh_\Lambda(M) \to \mu Sh_\Lambda(\Lambda)$, which is a generalized Poincaré-Lefschetz duality (which we expect to hold without swappable or fully stopped assumption).

Discussion of the result

- When Λ is not swappable or fully stopped, we construct examples where the microlocalization functor m_Λ is not spherical.
- The result on spherical functor and Serre duality is closely related to relative Calabi-Yau structure for the pair $m_\Lambda : Sh_\Lambda(M) \to \mu Sh_\Lambda(\Lambda)$, which is a generalized Poincaré-Lefschetz duality (which we expect to hold without swappable or fully stopped assumption).
- More generally, we can decompose Λ into smaller open pieces and consider microlocalization along these open pieces, which should correspond to further decomposing the Landau-Ginzburg model. We expect that one can prove spherical adjunction under certain relative swappability assumption.

Thank you!